BOUNDARY-VALUE PROBLEM FOR A TWO-DIMENSIONAL SECOND ORDER-TYPE EQUATION WITH DISCRETE ADDITIVE AND MULTIPLICATIVE DERIVATIVES


Summary. The present paper is concerned with the study of solutions to the boundary-value problem for a two-dimensional second order-type differential equation with a discrete additive derivative for one argument and a discrete multiplicative derivative for another argument.

We will determine the general solution of the considered equation, containing some derived sequences. Further, these unknown sequences are determined using an assigned boundary condition.

Keywords: Boundary-value problem, two-dimensional equation, discrete additive derivative, discrete multiplicative derivative, second order equations, general solutions of the equation, solutions of the boundary-value problem.

Introduction. As it is known, the boundary-value problem for an ordinary differential equation in both the continuous case [1] – [2] and the discrete case (the so-called finite-difference equation) is well studied [3] – [4]. The boundary-value problems for a differential equation with a multiplicative derivative in the continuous case was started almost a century ago [5], as for the discrete case, it was started in our papers [6] – [7]. We also study boundary-value problems for a second order-type differential equation with discrete additive-multiplicative and multiplicative-additive derivatives [8] – [9].

The present paper is concerned with the study of solutions to the boundary-value problem for a two-dimensional second order-type differential equation with a discrete additive derivative for one argument and a discrete multiplicative derivative for another argument.

Setting of the problem: Let’s consider the following boundary-value problem:

$$D_2^{[2]}D_1^{(1)}y_{mn} = f_{mn}, \quad 0 \leq m < M, \quad 0 \leq n < N, \quad (1)$$

$$y_{Mn} = a y_{0n} + \varphi_n, \quad 0 \leq n \leq N, \quad (2)$$

$$y_{mn} = b y_{m0} + \varphi_m, \quad 0 \leq m \leq M, \quad (3)$$

where $a$, $b$ are real constants, $\varphi_n, 0 \leq n \leq N$; $\varphi_m, 0 \leq m \leq M$ and $\varphi_m, 0 \leq m \leq M, 0 \leq n \leq N$ are preset real-valued sequences, $y_{mn}, 0 \leq m < M, 0 \leq n < N$ is the query sequence.

As it is known, the general solution of equation (1) is given by [10]:

$$y_{mn} = y_{0n} + \sum_{s=0}^{m-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{n-1} f_{sk}, \quad m \geq 1, \quad n \geq 1. \quad (4)$$

Substituting the general solution (4) to the boundary condition (2), we obtain:

$$y_{0n} + \sum_{s=0}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{n-1} f_{sk} = a y_{0n} + \varphi_n, \quad n \geq 0,$$

or given that

$$a \neq 1, \quad (5)$$

we have:

$$y_{0n} = \frac{\sum_{s=0}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{n-1} f_{sk} - \varphi_n}{a - 1}, \quad n \geq 0. \quad (6)$$
Then returning to the general solution (4) and substituting it to the boundary condition (3), we obtain:

\[ y_{0N} + \sum_{s=0}^{m-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} = b y_{m0} + \varphi_m, m \geq 0, \]

or taking into account (6), we have:

\[ \sum_{s=0}^{m-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} = \frac{\varphi_N}{a - 1} + \sum_{s=0}^{m-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} = b y_{m0} + \varphi_m, m \geq 0, \]

or, after simplification, we obtain:

\[ \sum_{y=m}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} + a \sum_{s=0}^{m-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} = (a - 1) b y_{00} + (a - 1) \varphi_0 + \varphi_N, m \geq 0. \tag{7} \]

Giving \( m \) a value starting from zero, we obtain:

when \( m = 0 \):

\[ \sum_{s=0}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} = (a - 1) b y_{00} + (a - 1) \varphi_0 + \varphi_N, \tag{8} \]

when \( m = 1 \):

\[ \sum_{s=1}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} + a D_1^{(1)} y_{00} \prod_{k=0}^{N-1} f_{0k} = (a - 1) b y_{10} + (a - 1) \varphi_1 + \varphi_N, \tag{9} \]

Subtracting formula (8) from (9), we get:

\[ -D_1^{(1)} y_{00} \prod_{k=0}^{N-1} f_{0k} + a D_1^{(1)} y_{00} \prod_{k=0}^{N-1} f_{0k} = (a - 1) b (y_{10} - y_{00}) + (a - 1) (\varphi_1 - \varphi_2), \]

after reducing by \((a - 1)\), we have:

\[ D_1^{(1)} y_{00} \prod_{k=0}^{N-1} f_{0k} = b D_1^{(1)} y_{00} + (\varphi_1 - \varphi_0), \]

from which under the conditions of

\[ \prod_{k=0}^{N-1} f_{0k} \neq b, \tag{10} \]

we obtain:

\[ D_1^{(1)} y_{00} = y_{10} - y_{00} = \frac{\varphi_1 - \varphi_0}{\prod_{k=0}^{N-1} f_{0k} - b}. \tag{11} \]

When \( m = 2 \) of (7), we get:

\[ \sum_{s=2}^{M-1} (D_1^{(1)} y_{s0}) \prod_{k=0}^{N-1} f_{sk} + a D_1^{(1)} y_{00} \prod_{k=0}^{N-1} f_{0k} + a D_1^{(1)} y_{10} \prod_{k=0}^{N-1} f_{1k} \prod_{k=0}^{N-1} f_{sk} = (a - 1) b y_{20} + (a - 1) \varphi_2 + \varphi_N. \tag{12} \]

If we subtract formula (9) from (12), we have:

\[ -D_1^{(1)} y_{10} \prod_{k=0}^{N-1} f_{1k} + a D_1^{(1)} y_{10} \prod_{k=0}^{N-1} f_{1k} = (a - 1) b (y_{20} - y_{10}) + (a - 1) (\varphi_2 - \varphi_1), \]
reducing by \((a-1)\), we get:

\[
D^{(1)}_1 y_{10} \prod_{k=0}^{N-1} f_{1k} = b D^{(1)}_1 y_{10} + \varphi_2 - \varphi_1
\]

or under the condition of

\[
\prod_{k=0}^{N-1} f_{1k} \neq b, \quad (13)
\]

we have:

\[
D^{(1)}_1 y_{10} = y_{20} - y_{10} = \frac{\varphi_2 - \varphi_1}{\prod_{k=0}^{N-1} f_{1k} - b}, \quad (14)
\]

Continuing this process under the conditions

\[
\prod_{k=0}^{N-1} f_{yk} \neq b, \quad s \geq 0, \quad (15)
\]

we get

\[
D^{(1)}_1 y_{s0} = y_{s+10} - y_{s0}, \quad s \geq 0. \quad (16)
\]

Then \(\varphi_{0n}\) is obtained from the event (6).

Thus, we get the following statements:

**Theorem:** If \(a, b\) are real numbers, \(\varphi_m, 0 \leq n \leq N; \varphi_m, 0 \leq m \leq M\) and \(f_{mn}, 0 \leq m < M, 0 \leq n < N\) are preset real-valued sequences, then under the condition (5) and (15) there is a solution to the boundary-value problem (1) – (3), represented as (4) where

\[
D^{(1)}_1 y_{s0} = \frac{\varphi_{s+1} - \varphi_s}{\prod_{k=0}^{N-1} f_{sk} - b}, \quad s \geq 0, \text{ and } y_{0n} \text{ is determined by the formula (6).}
\]

**Literature**

6. Fryazinov I.V., Bakarova M.I. On economic difference schemes to solve the heat-transfer equations in efficient, cylindrical and spatial polar coordinates. JVM and MF, No.9, 1972, pp. 352 – 363.